EVERY SEPARABLE METRIC SPACE IS LIPSCHITZ EQUIVALENT TO A SUBSET OF c_0^+

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ABSTRACT

It is shown that there is a constant K so that, for every separable metric space X, there is a map $T: X \to c_0$ satisfying

$$d(x,y) \leqq \|Tx - Ty\| \leqq Kd(x,y) \text{ for every } x, y \in X.$$

Two metric spaces X, Y are called Lipschitz equivalent if there is a map T from X onto Y, such that T and T^{-1} satisfy a Lipschitz condition. The spaces are called uniformly equivalent if there is a map T from X onto Y, such that T and T^{-1} are uniformly continuous.

Only little is known at present on the uniform or Lipschitz classification of Banach spaces. It is, for example, an open problem whether there are two nonisomorphic Banach spaces which are Lipschitz equivalent, ([3], p. 89) or uniformly equivalent ([5], pp. 162–163 and [2], p. 283).

A related problem is the question whether the existence of a Lipschitz (resp. uniform) homcomorphism from a Banach space X into a Banach space Y, implies that there is a linear isomorphism from X into Y. Mankiewicz [10] has shown that for Lipschitz embeddings, the answer is positive if Y is a separable conjugate space (or more generally if Y has the Radon-Nikodym property). We prove here that the answer is negative if $Y = c_0$. We show that every separable Banach space is Lipschitz equivalent to a subset of c_0. It is well known that not every such space is isomorphic to a subspace of c_0.

Our result is of interest also in the context of the general theory of metric spaces. It is known that every separable metric space is isometric to a subset of $C(0,1)$. ([1], p. 185). Thus our result implies that every separable metric space is Lipschitz equivalent to a subset of c_0.

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Let X be a Banach space with a normalized basis $\{e_i\}_{i=1}^{\infty}$ (i.e. such that $\|e_i\| = 1$ for all i). For every $x \in X$ there is a unique representation $x = \sum_{i=1}^{\infty} x_i e_i$. We denote by $\{P_n\}_{n=0}^{\infty}$ the partial sum operators (i.e. $P_n x = \sum_{i=1}^{n} x_i e_i$, $P_0 x = 0$). It is known that for such a space X, there is an equivalent norm (namely $|||x||| = \sup_{m<n} \|P_n x - P_m x\|$) in which all the operators $P_n - P_m$ are of norm 1. In the sequel we shall assume, therefore, that $\|P_n - P_m\| = 1$ $0 \leqq m, n < \infty$. It follows from the convexity of the norm that for every $x = \sum_{i=1}^{\infty} x_i e_i \in X$, all natural numbers $m < n$ and real numbers $0 \leqq t, s \leqq 1$

$$\| t x_m e_m + P_n x - P_m x + s x_{n+1} e_{n+1} \| \leqq \| x \| . \tag{1}$$

Definition 1. An element $x \in X$, for which there is an integer n satisfying $P_n x = x$, is called finitely supported. The minimal such n is called the length of x, and is denoted by $l(x)$.

Definition 2. Let $0 \neq x = \sum_{i=1}^{\infty} x_i e_i \in X$, and let $0 < \beta < \| x \|$. Let n be the first integer such that $\|x - P_n x\| < \beta$ (i.e. $\|x - P_{n-1} x\| \geqq \beta$). Let $0 \leqq t \leqq 1$ be such that $\|x - P_n x + t x_n e_n\| = \beta$. We define the map Q_β by

$$Q_\beta x = P_n x - t x_n e_n = P_{n-1} x + (1-t) x_n e_n. \tag{2}$$

From the definition we get immediately that:

a) $Q_\beta x$ is finitely supported,

$$\|Q_\beta x\| \leqq \|x\| \quad \text{(by (1))} \quad \text{and} \quad \|x - Q_\beta x\| = \beta. \tag{3}$$

b) For every $y \in X$ satisfying $l(y) < l(Q_\beta x)$, we have

$$\| x - y \| \geqq \| x - Q_\beta x \| = \beta . \tag{4}$$

Lemma 1. *Let $a > 0$, $\alpha > 0$ be real numbers. Then there is a sequence $\{z^j(a,\alpha)\}_{j=1}^{\infty}$ of finitely supported elements in X satisfying:*

a) *For every j, $\|z^j(a,\alpha)\| \leqq \alpha$.*

b) *$l(z^j(a,\alpha))$ is a (not necessarily strictly) increasing and unbounded function of j.*

c) *For every finitely supported $z \in X$ satisfying $\|z\| \leqq \alpha$, there is a j such that $\|z - z^j(a,\alpha)\| < a$, $l(z^j(a,\alpha)) = l(z)$ and for $m = l(z)$, $(z^j(a,\alpha))_m z_m > 0$.*

(We denote by z_m the coefficient of e_m in the expansion of z).

Proof. For every integer n, the sets

$$A_n = \{z \in X \colon l(z) = n,\ z_n > 0,\ \|z\| \leqq \alpha\}$$

and

$$B_n = \{z \in X \colon l(z) = n,\ z_n < 0,\ \|z\| \leqq \alpha\}$$

are totally bounded, hence there are finite a-nets in A_n and in B_n denoted by $\{z^j(a,\alpha)\}_{j=p_{2n-2}+1}^{p_{2n-1}}$ and $\{z^j(a,\alpha)\}_{j=p_{2n-1}+1}^{p_{2n}}$, respectively (where $\{p_n\}_{n=1}^{\infty}$ is an increasing sequence of integers, and $p_0 = 0$). The sequence $\{z^j(a,\alpha)\}_{j=1}^{\infty}$ satisfies all the requirements in the statement of the lemma.

Definition 3. Let $a > 0$ be a real number. Let $n \geqq 6$, $1 \leqq k \leqq 2n/3$, and j be natural numbers. Denote by i the triple $i = (n,k,j)$. We define

$$M_i^a = \{x \in X : (n-2)a \leqq \|x\| \leqq (n+1)a,\ Q_{ka}x = z^j(a,na)\}$$

$$t_i^a = (k-1)a$$

where $\{z^j(a,na)\}_{j=1}^{\infty}$ is the sequence constructed in lemma 1.

Remark. We may have $M_i^a = \varnothing$ for some i. In the sequel we shall consider only the non-empty M_i^a. In order not to complicate the notation any more, we do not use a special notation for them.

Lemma 2. *For every $x \in X$ there is only a finite number of triples $i = (n,k,j)$ for which $d(x, M_i^a) < t_i^a$.*

Proof. For $n > 3(\|x\|/a + 1)$, every $1 \leqq k \leqq 2n/3$, every $j \in N$ and every $y \in M_i^a$ we have

$$\|y - x\| \geqq \|y\| - \|x\| \geqq (n-2)a - (n/3 - 1)a = 2na/3 - a \geqq ka - a = t_i^a.$$

Hence, $d(x, M_i^a) \geqq t_i^a$ for such i.

Assume now that $n < 3(\|x\|/a + 1)$. We have $\|x\| > (n-3)a/3 \geqq a$. Since $Q_{a/2}x$ is finitely supported, there is a j_0 such that $l(z^{j_0}(a,na)) > l(Q_{a/2}x)$. Hence for every $j \geqq j_0$ we have $l(z^j(a,na)) > l(Q_{a/2}x)$. For such j, for $i = (n,k,j)$ and for every $y \in M_i^a$ we have

$$\|y - x\| \geqq \|y - Q_{a/2}x\| - \|x - Q_{a/2}x\| \geqq \|y - Q_{ka}y\| - a/2$$

$$= ka - a/2 > (k-1)a = t_i^a.$$

(The second inequality follows by (4)). Hence we have $d(x, M_i^a) \geqq t_i^a$ also for such i, and the lemma is proved.

Lemma 3. *Let $a > 0$ and $x, y \in X$ satisfy $\|x\| \geqq \|y\|$ and $\|x - y\| \geqq 36a$. Then there is a triple $i = (n,k,j)$ for which*

a) $d(x, M_i^a) < a$, $d(y, M_i^a) \geqq t_i^a$

b) $t_i^a \leqq \|x - y\|/2$

c) $t_i^a - a \geqq \|x - y\|/4$.

PROOF. Let

$$n = [\|x\|/a] + 1$$

and

$$k = [\|x - y\|/3a]$$

(where $[t]$ denotes the largest integer $\leqq t$). We have that

$$2na \geqq 2\|x\| \geqq \|x - y\|$$

hence

$$k \leqq 2n/3$$

and

$$\|Q_{ka}x\| \leqq \|x\| \leqq na \ .$$

Let $z^j = z^j(a, na)$ be the element ensured by part c of Lemma 1, corresponding to $z = Q_{ka}x$, and let $i = (n, k, j)$. We prove that this triple i has all the desired properties. Put

$$u = z^j + x - Q_{ka}x.$$

Then

$$\|u - x\| = \|z^j - Q_{ka}x\| < a$$

and

$$Q_{ka}u = z^j$$

(By part c of Lemma 1). We have

$$(n-2)a = (n-1)a - a \leqq \|x\| - a \leqq \|u\| \leqq \|x\| + a \leqq na + a = (n+1)a.$$

Hence $u \in M_i^a$ and $d(x, M_i^a) \leqq \|u - x\| < a$.

On the other hand, for every $v \in M_i^a$ we have

$$Q_{ka}v = z^j.$$

Hence

$$\|y - v\| \geqq \|y - x\| - \|x - Q_{ka}x\| - \|Q_{ka}x - z^j\| - \|z^j - v\|$$

$$\geqq 3ka - ka - a - ka = (k-1)a = t_i^a.$$

Hence $d(y, M_i^a) \geqq t_i^a$. This proves part (a) of the lemma. Part (b) of the lemma holds since

$$t_i^a = (k-1)a \leqq \|x-y\|/3 - a \leqq \|x-y\|/2.$$

Finally, part (c) holds since

$$(t_i^a - a)/\|x-y\| \geqq (k-2)a/3(k+1)a \geqq 1/4$$

(Observe that $k > \|x-y\|/3a - 1 \geqq 36/3 - 1 = 11$).

We are now ready to prove our main result.

THEOREM. *There is a constant $K > 0$ such that for every separable metric space X, there is a map $T: X \to c_0$ satisfying*

$$d(x,y) \leqq \|Tx - Ty\| \leqq Kd(x,y)$$

for every $x, y \in X$.

PROOF. By the theorem of Banach and Mazur, every separable metric space is isometric to a subset of $C(0,1)$, ([1], pp. 185–188), which is a Banach space with a basis. We assume, therefore, that X is a Banach space with a normalized basis, satisfying $\|P_n - P_m\| = 1$ $0 \leqq m < n < \infty$.

For every $a > 0$ we denote by T_a the map $T_a: X \to c_0$ defined by

$$(T_a x)_i = \max(0, t_i^a - d(x, M_i^a)).$$

By Lemma 2, $T_a x$ is finitely supported for every $x \in X$. Clearly $T_a 0 = 0$ and $\|T_a x - T_a y\| \leqq \|x - y\|$ for all $x, y \in X$. If $\|x\| \geqq \|y\|$ and $\|x - y\| \geqq 36a$, then by Lemma 3 there is an i such that

$$(T_a y)_i = 0 \qquad t_i^a - a \leqq (T_a x)_i \leqq t_i^a$$

and hence

$$|(T_a x)_i - (T_a y)_i| \geqq t_i^a - a \geqq \|x - y\|/4.$$

We define next a map $S_a: X \to c_0$ by

$$(S_a x)_i = \min((T_a x)_i, 72a).$$

The map S_a has the following properties: $S_a 0 = 0$; $S_a x$ is finitely supported and $\|S_a x\| \leqq 72a$ for every $x \in X$. Also $\|S_a x - S_a y\| \leqq \|x - y\|$ for every $x, y \in X$, and if $36a \leqq \|x - y\| \leqq 72a$ then $\|S_a x - S_a y\| \geqq \|x - y\|/4$.

Define now

$$T: X \to \left(\sum_{n=1}^{\infty} \oplus c_0\right)_0 \cong c_0$$

by

$$T = 4\left(T_1 \oplus \sum_{n=1}^{\infty} \oplus S_{1/2^n} \right) .$$

i.e.,

$$Tx = 4(T_1x, S_{1/2}x, \cdots, S_{1/2^n}x, \cdots).$$

This map has all the desired properties.

REMARK. We can prove that one can take any $K > 6$. (Our proof does not ensure $K = 4$ since we have also to renorm $C(0, 1)$ so that for some Schauder basis $\|P_n - P_m\| \leqq 1$ for all $m < n$). We do not know the best possible constant. We know however that $K \geqq 2$. This is shown by Proposition 3 below.

If we replace the requirement that T is a Lipschitz equivalence, by the requirement that T is only a uniform equivalence, then a slight modification of the proof of the theorem will show the existence of a map T so that $T(X)$ is even bounded. It turns out however that this can be done directly in a very simple manner by the following observation.

PROPOSITION 1. *A separable infinite-dimensional $C(K)$ space (K compact) is uniformly equivalent to a bounded subset of itself.*

PROOF. For every $f \in C(K)$ let $\phi(f) \in C(K)$ be defined by:

$$\phi f(x) = \max\{-2, \min\{0, f(x)\}\}.$$

Define now

$$T : C(K) \to \left(\sum_{n=0}^{\infty} \oplus C(K)\right) \oplus \left(\sum_{n=0}^{\infty} \oplus C(K)\right) \cong C(K)$$

by

$$Tf = \sum_{n=0}^{\infty} \oplus \phi(n+f) \oplus \sum_{n=0}^{\infty} \oplus \phi(n-f).$$

An easy computation shows that for every $f, g \in C(K)$

$$\min\{1, \|f-g\|\} \leqq \|Tf - Tg\| \leqq \|f-g\|, \quad \|Tf\| \leqq 3.$$

Hence T defines the desired equivalence.

COROLLARY. *Every separable metric space is uniformly equivalent to a bounded subset of c_0.*

Proposition 1 is of interest in connection with a problem raised by Gorin [6] whether l_2 is uniformly equivalent to a bounded subset of itself.

Besides c_0 there is another important and well known example of a Banach space which fails to have the Radon-Nikodym property, namely $L_1(0, 1)$. It is therefore natural to ask whether it could replace c_0 in the theorem above. The answer is negative. Indeed we prove a stronger result.

PROPOSITION 2. *The space c_0 is not uniformly equivalent to a subset of $L_p(0, 1)$ $1 \leqq p < \infty$.*

PROOF. Assume to the contrary that c_0 is uniformly equivalent to a subset of $L_p(0, 1)$ $1 \leqq p < \infty$. By [4] (p. 48), a uniformly continuous map on c_0 maps bounded sets into bounded sets. Using Corollary 1, we get that every separable metric space is uniformly equivalent to a bounded subset of $L_p(0, 1)$, which in turn is uniformly equivalent to a bounded subset of $L_2(0, 1)$ [9]. This contradicts a result of Enflo [5] who exhibited a separable metric space which is not uniformly equivalent to a subset of l_2.

We return now to the Lipschitz constant appearing in the theorem.

PROPOSITION 3. *Let $T: l_1 \to c_0$ satisfy $\|x - y\| \leqq \|Tx - Ty\| \leqq K\|x - y\|$ for every $x, y \in l_1$. Then $K \geqq 2$.*

PROOF. Whithout loss of generality we may assume that $T0 = 0$. Assume that $K < 2$ and hence $4 - 2K > 0$. Denote by $\{e_i\}_{i=1}^{\infty}$ the unit vector basis in l_1 and let

$$M = \{n \in N : |(Te_1)_n - (Te_2)_n| \geqq 4 - 2K\} .$$

For every $i, j \geqq 3$, $i \neq j$ define

$$M_{i,j} = \{n \in N : |(Te_i)_n - (Te_j)_n| \geqq 4 - 2K\}.$$

M and $M_{i,j}$ are finite sets. We show that for every $i, j \geqq 3, i \neq j, M \cap M_{i,j} \neq \emptyset$.

Indeed,

$$\|T(e_1 + e_i) - T(e_2 + e_j)\| \geqq \|(e_1 + e_i) - (e_2 + e_j)\| = 4$$

Hence there is an $n_0 \in N$ for which

$$|T(e_1 + e_i)_{n_0} - T(e_2 + e_j)_{n_0}| \geqq 4$$

Consequently,

$$|(Te_1)_{n_0} - (Te_2)_{n_0}| \geqq |T(e_1 + e_i)_{n_0} - T(e_2 + e_j)_{n_0}|$$

$$-|T(e_1+e_i)_{n_0}-(Te_1)_{n_0}|-|T(e_2+e_j)_{n_0}-(Te_2)_{n_0}|$$

$$\geqq 4-\|T(e_1+e_i)-Te_1\|-\|T(e_2+e_j)-Te_2\|\geqq 4-K-K=4-2K,$$

hence $n_0 \in M$. The same reasoning shows also that $n_0 \in M_{i,j}$, and hence $M \cap M_{i,j} \neq \emptyset$.

Denote by P_M the natural projection of c_0 on $\operatorname{span}\{e_n\}_{n\in M}$. Then $\{P_M Te_i\}_{i=1}^{\infty}$ is a bounded set in a finite dimensional space. This contradicts the fact that the distance between any two of its members is $\geqq 4-2K$.

We conclude this note by mentioning a problem which arises naturally from our work.

PROBLEM. Is $C(0,1)$ Lipschitz equivalent to c_0? (This is part of the problem of [3] mentioned in the introduction.) In connection with this problem, we mention that J. Lindenstrauss proved in [7] (pp. 273–275) that from every metric space containing a $C(K)$ space (K — compact metric space), there is a Lipschitz projection on $C(K)$. Thus, if T is a Lipschitz embedding of $C(0,1)$ into c_0, then there is a Lipschitz projection from c_0 onto $TC(0,1)$.

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